Variational Principles for Some Nonlinear Wave Equations

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Using the semi-inverse method proposed by Ji-Huan He, variational principles are established for some nonlinear wave equations arising in physics, including the Pochhammer-Chree equation, Zakharov-Kuznetsov equation, Korteweg-de Vries equation, Zhiber-Shabat equation, Kawahara equation, and Boussinesq equation.

Key words: Variational Theory; Semi-Inverse Method; Nonlinear Equation.
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1. Introduction

Generally speaking, there exist two basic ways to describe a physical problem [1]: (1) by differential equations (DE) with boundary or initial conditions; (2) by variational principles (VP). The VP model has many advantages over its DE partner: simple and compact in form while comprehensive in content, encompassing implicitly almost all information characterizing the problem under consideration [1 – 5]. Variational methods have been, and continue to be, popular tools for nonlinear analysis. When contrasted with other approximate analytical methods, variational methods combine the following two advantages: (1) they provide physical insight into the nature of the solution of the problem; (2) the obtained solutions are the best among all the possible trial functions. The variational-based analytical methods, e. g., the variational iteration method [6 – 11] and He's variational method [1], have become hot topics in recent publications. Although the variational principles of fluid dynamics [1 – 5] have been studied for a long time, yet the general variational principles of various nonlinear wave equations have not been dealt with systematically.

In this paper we illustrate how to establish a variational formulation for a nonlinear problem using the semi-inverse method proposed by Ji-Huan He [2].

2. Variational Formulations

In recent years, variational principles in physics have resulted in a great amount of research. Xu [12], Öziş and Yıldırım [13] established variational principles for the Schrödinger equation. Zhang [14] found a variational principle for the Zakharov equation. Liu et al. [15] constructed a variational formulation for the Ginzburg-Landau equation. Wang [16] suggested a variational theory for the variable coefficients Korteweg-de Vries (KdV) equation. Zhou [17, 18] studied variational principles for the physiological flow and the Broer-Kaup-Kupershmidt equation. Tao [19] established the variational formulation of the inviscid compressible fluid. Wu [20] obtained a variational formulation for higher-order water-wave equations. In [21] Wazwaz concluded that the $K(m,n)$ equations could not be derived from a first-order Lagrangian except for $m = n = 1$. Xu [22] first pointed out that Wazwaz's conclusion is not entirely correct, using the semi-inverse method; Xu succeed in establishing the needed variational principle for $K(m,n)$ equations.

In this paper we will illustrate how to establish a variational principle for a nonlinear problem using the semi-inverse method.

2.1. The Pochhammer-Chree Equation

Consider the Pochhammer-Chree equation governed by [23]

$$u_{tt} - u_{ttxx} - (\alpha u + \beta u^{n+1} + \gamma u^{2n+1})_{xx} = 0,\ n \geq 1. \quad (1)$$

We introduce a special function Φ defined as

$$\Phi_{xx} = u, \quad (2)$$

$$\Phi_{tt} = u_{tt} + \alpha u + \beta u^{n+1} + \gamma u^{2n+1}, \quad (3)$$

so that (1) is automatically satisfied. We will apply the semi-inverse method [2–5] to search for the needed variational formulation:

$$J(u,\Phi)=\int\int L\,\mathrm{d}x\mathrm{d}t. \tag{4}$$

Here L is a trial Lagrangian defined by

$$L=u\Phi_{tt}-(u_{tt}+\alpha u+\beta u^{n+1}+\gamma u^{2n+1})\Phi_{xx} + F(u), \tag{5}$$

where F is a unknown function of u and/or its derivatives. The merit of the trial Lagrangian is that the stationary condition with respect to Φ leads to (1).

Now the stationary condition with respect to u is

$$\Phi_{tt}-\{\Phi_{xxtt}+\alpha\Phi_{xx}+(n+1)\beta u^n\Phi_{xx} + (2n+1)\gamma u^{2n}\Phi_{xx}\}+\frac{\delta F}{\delta u}=0, \tag{6}$$

where $\delta F/\delta u$ is called He's variational derivative with respect to u.

Concerning (2) and (3), we set

$$\frac{\delta F}{\delta u}=-\Phi_{tt}+\{\Phi_{xxtt}+\alpha\Phi_{xx}+(n+1)\beta u^n\Phi_{xx} +(2n+1)\gamma u^{2n}\Phi_{xx}\} =n\beta u^{n+1}+2n\gamma u^{2n+1}. \tag{7}$$

So, the unknown F can be determined as

$$F=\frac{n}{n+2}\beta u^{n+2}+\frac{n}{n+1}\gamma u^{2n+2}. \tag{8}$$

We, therefore, obtain the following variational formulation:

$$J(u,\Phi)=\int\int\Big\{u\Phi_{tt}-\big(u_{tt}+\alpha u+\beta u^{n+1} +\gamma u^{2n+1}\big)\Phi_{xx}+\frac{n}{n+2}\beta u^{n+2} +\frac{n}{n+1}\gamma u^{2n+2}\Big\}\mathrm{d}x\mathrm{d}t. \tag{9}$$

2.2. The Modified Zakharov-Kuznetsov Equation

Consider the generalized form of the modified Zakharov-Kuznetsov (ZK) equation [24]

$$u_t+au^{n/2}u_x+b(u_{xx}+u_{yy})_x=0,\quad n\geq 1. \tag{10}$$

We introduce a special function Ψ defined as

$$\Psi_x=-u, \tag{11}$$

$$\Psi_t=\frac{2a}{n+2}u^{(n+2)/2}+b(u_{xx}+u_{yy}). \tag{12}$$

Therefore, (10) is automatically satisfied. In view of the semi-inverse method [2–5], we construct a trial functional in the form

$$J(u,\Phi)=\int\int L\,\mathrm{d}x\mathrm{d}t. \tag{13}$$

Here L is a trial Lagrangian defined by

$$L=u\Psi_t+\left\{\frac{2a}{n+2}u^{(n+2)/2}+b(u_{xx}+u_{yy})\right\}\Psi_x + F(u), \tag{14}$$

where F is a unknown function of u and/or its derivatives. The merit of the trial Lagrangian is that the stationary condition with respect to Ψ leads to (10).

Here, the stationary condition with respect to u is

$$\Psi_t+au^{n/2}\Psi_x+b(\Psi_{xxx}+\Psi_{xyy})+\frac{\delta F}{\delta u}=0. \tag{15}$$

Concerning (11) and (12), we set

$$\frac{\delta F}{\delta u}=\left(a-\frac{2a}{n+2}\right)u^{(n+2)/2}. \tag{16}$$

Thus the unknown F can be determined as

$$F=\frac{2}{n+4}\left(a-\frac{2a}{n+2}\right)u^{(n+4)/2}. \tag{17}$$

The needed variational formulation is obtained:

$$J(u,\Psi)=\int\int\Big\{u\Psi_t+\Big[\frac{2a}{n+2}u^{(n+2)/2} + b(u_{xx}+u_{yy})\Big]\Psi_x + \frac{2}{n+4}(a-\frac{2a}{n+2})u^{(n+4)/2}\Big\}\mathrm{d}x\mathrm{d}t. \tag{18}$$

2.3. The Generalized ZK Equation

Let us concern the generalized ZK equation [24]

$$u_t+au^nu_x+b(u_{xx}+u_{yy})_x=0,\quad n\geq 1. \tag{19}$$

With the same manipulation as illustrated before, we obtain the variational formulation

$$J(u,H) = \int\int\left\{uH_t + \left[\frac{a}{n+1}u^{n+1} + b(u_{xx}+u_{yy})\right]H_x + \frac{1}{n+2}\left(a - \frac{a}{n+1}\right)u^{n+2}\right\}\mathrm{d}x\mathrm{d}t, \quad (20)$$

where H is a special function determined as

$$H_x = -u, \quad (21)$$

$$H_t = \frac{a}{n+1}u^{n+1} + b(u_{xx}+u_{yy}). \quad (22)$$

2.4. The Modified KdV Equation

Consider a modified KdV equation [24]

$$u_t + (1+bu^{1/2})u_x + \frac{1}{2}u_{xxx} = 0. \quad (23)$$

Let R be a special function defined as

$$R_x = -u, \quad (24)$$

$$R_t = u + \frac{2}{3}bu^{3/2} + \frac{1}{2}u_{xx}. \quad (25)$$

By the same operation as above, we arrive at

$$J(u,R) = \int\int\left\{uR_t + \left[u + \frac{2}{3}bu^{3/2}\right]R_x + \frac{1}{2}u_{xx}R_x + \frac{2}{15}bu^{5/2} - \frac{1}{4}(u_x)^2\right\}\mathrm{d}x\mathrm{d}t. \quad (26)$$

2.5. The Zhiber-Shabat Equation

The Zhiber-Shabat equation reads [25]

$$u_{xt} + pe^u + qe^{-u} + re^{-2u} = 0. \quad (27)$$

Applying the semi-inverse method, we obtain with ease the variational formulation

$$J(u) = \int\int\left\{-\frac{1}{2}u_xu_t + pe^u - qe^{-u} - \frac{r}{2}e^{-2u}\right\}\mathrm{d}x\mathrm{d}t. \quad (28)$$

2.6. The Modified Kawahara Equation

For the modified Kawahara equation [26]

$$u_t + au^2u_x + bu_{xxx} - ku_{xxxxx} = 0, \quad (29)$$

its variational formulation can be obtained by the same way as before, which reads

$$J(u,M) = \int\int\left\{uM_t + \left(\frac{a}{3}u^3 + bu_{xx} - ku_{xxxx}\right)M_x + \frac{a}{6}u^4\right\}\mathrm{d}x\mathrm{d}t, \quad (30)$$

where M is a special function defined by

$$M_x = -u, \quad (31)$$

$$M_t = \frac{a}{3}u^3 + bu_{xx} - ku_{xxxx}. \quad (32)$$

2.7. The Combined KdV-modified KdV Equation

The combined KdV-modified KdV equation can be written in the form [27]

$$u_t + puu_x + qu^2u_x + u_{xxx} = 0. \quad (33)$$

By the semi-inverse method, the functional reads

$$J(u,N) = \int\int\left\{uN_t + \left(\frac{p}{2}u^2 + \frac{q}{3}u^3 + u_{xx}\right)N_x + \frac{p}{6}u^3 + \frac{q}{6}u^4\right\}\mathrm{d}x\mathrm{d}t, \quad (34)$$

where N is a special function defined by

$$N_x = -u, \quad (35)$$

$$N_t = \frac{p}{2}u^2 + \frac{q}{3}u^3 + u_{xx}. \quad (36)$$

2.8. The Cubic Boussinesq Equation

The cubic Boussinesq equation is [28]

$$u_{tt} - u_{xx} + (2u^3)_{xx} - u_{xxxx} = 0. \quad (37)$$

The variational formulation reads

$$J(u,\Gamma) = \int\int\left\{u\Gamma_{tt} + (3u^3 - u - u_{xx})\Gamma_{xx} + \frac{3}{2}u^4\right\}\mathrm{d}x\mathrm{d}t, \quad (38)$$

where Γ is a special function defined by

$$\Gamma_{xx} = -u, \quad (39)$$

$$\Gamma_{tt} = 3u^3 - u - u_{xx}. \quad (40)$$

2.9. The Fourth-Order Boussinesq Equation

The fourth-order Boussinesq equation is [28]

$$u_{tt} - a^2 u_{xx} - b(u^2)_{xx} + u_{xxxx} = 0. \quad (41)$$

The variational formulation turns out to be

$$J(u, \Omega) = \int\int \left\{ u\Omega_{tt} + (u_{xx} - a^2 u - bu^2)\Omega_{xx} - \frac{b}{3}u^3 \right\} \mathrm{d}x\mathrm{d}t, \quad (42)$$

where Ω is a special function defined by

$$\Omega_{xx} = -u, \quad (43)$$

$$\Omega_{tt} = u_{xx} - a_2 u - bu_2. \quad (44)$$

3. Conclusion

We obtained the variational formulations for the discussed equations. Variational-based analytical methods have been shown exceedingly elegant and effective in solving nonlinear problems [6–11], and the variational-based finite element method [29, 30] has been, and continues to be, a popular numerical tool. The results obtained in this paper might find some potential applications in engineering.

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